Historical and Mathematical Overview of Einstein's General Relativity

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Description of Work

Newton's physics was the widely accepted way to interpret movement of bodies, and the action of gravitational force. However, observations showed that his model was flawed. His model claimed that gravity was the result of attractive forces between objects, but this was not the case, and was especially problematic for large-scale computations. Einstein conceptualized the theory of special relativity, which described the relationship between space and time, and described motion in terms of inertial frames of reference. He was able to make corrections to Newton's theory, and a more accurate model of motion when it approaches light speed. This was still a simplified model of reality. In general relativity, Einstein takes gravity into account and shows how gravitational action is actually the warping of space and time caused by massive objects. Einstein used Riemannian and differential geometry to develop his field equations, which describe the theory of general relativity. His theory helps to describe the motion of planetary bodies, black holes, time dilation, and is very important for modern astrophysics.

This work will present the motivation for general relativity, applications of the theory, and the Schwarzschild solution for the path of a photon in the presence of a single massive object.

Contents

1 Historical Background

Throughout time, people have looked up at the heavens and tried to understand the motion of the planets and stars. Many models were proposed to describe earthly and heavenly motion. Each model explained earthly and heavenly motion differently, with changing ideas about the nature of matter and referential frames. Each model had its imperfections and did not correctly explain or predict observed motion. Einstein's theory of general relativity solved many of these problems.

The first ideas about physics are known as the physics of common sense, because they are based on intuition and observation. [1, p. 11] Aristotle claimed that since the shape of the shadow by the earth on the moon is roughly circular, the earth must be spherical. [1, p. 12] He believed that there were four earthly elements, and each object had its own properties that governed its motion. For example, heavy objects naturally move downward and light objects naturally move upward. Unless acted on by another force, heavy or light objects would move in their natural way, towards or away from the center of the earth. [1, p. 13] Since arrows or canon balls are initially acted on by another force, they may move perpendicular to the earth for a while, until they are pulled to the earth's surface.

Aristotle believed that the heavenly bodies, which move around the earth in circles, are not made of the same elements of earthly bodies. They are composed of *aether*, which naturally moves in circular motion, thus explaining the circular motion of heavenly bodies. [1, p. 14] It was believed, though, that the earth remained still, and did not move; it did not even rotate about its axis. [1, p. 15]

The Ptolemaic system was a flexible, and complex model of motion. He proposed that there need not be uniform circular motion. Earth could be set off-center in the circle about which a planet rotates, which thus would account for the retrograde and variable speed motion of planets. [1, p. 28] In order to account for the apparent "wandering" of the planets, Ptolemy's system included a complicated combination of circles and loops that described how planets moved in relation to one another. [1, p. 29-31]

Copernicus's system initially greatly resembled Ptolemy's. However, he placed the sun, instead of the Earth, at the center of the universe and proposed that the Earth and other planets moved about the sun. The apparent motion of the sun, moon, stars, and other planets, could be explained by their rotation at different speeds about the sun and the Earth's daily axial rotation. [1, p. 35] Ptolemy had to make arbitrary assumptions to account for the observed motion of Mercury and Venus, but Copernicus's system simplified this. The orbits of these two planets had to reside within the distance between the sun and Earth, and the retrograde motion was easily explained. [1, p. 38] His assumptions allowed distances between the planets, the scale of the solar system, and the planets' time of revolution to be computed. Copernicus's ideas about gravity, such as how the air around the earth was somehow attached to the earth, and that each planet kept its spherical shape due to gravitational cohesion, were not very developed, but helped to develop future theories about universal gravitation and inertia. [1, p. 48]

Galileo looked up at the heavens with a telescope and made discoveries that overthrew established notions about the world. Aristotle held that the heavens were unchangeable, change could only happen on earth. However, Galileo discovered a new star in the constellation Serpentarius, that had not been observed before, thus proving that change occurred in the heavens. [1, p. 55] He also observed that the moon was not perfectly smooth and spherical, but had ridges and mountains, similar to the surface of the earth. Thus, the ancients were wrong in their belief that the heavenly bodies were perfect, and the earth was unique. Like the other planets, the earth shines, and the planets shine from light reflected from the sun. [1, p. 64] He also conjectured that the stars must be located at great distances compared to the other planets, since, when seen through a telescope, the planets are magnified to look like discs, but the stars are not. [1, p. 64] Galileo's discoveries proved that Aristotelian and Ptolemaic systems did not correctly describe the universe. The old physics had to be overthrown, and a new physics, that supported the Copernican system, had to be established. [1, p. 78]

The universality of motion became the accepted view of motion in the seventeenth century. Motion occurred everywhere, and it was the same kind of motion everywhere. It was believed that motion did not depend on location or scale of the object. The laws of motion as observed on Earth should be applicable to motion of celestial bodies as well. [2, p. 37] The new laws of motion depended on the idea that nature was uniform everywhere. Kepler expanded on Aristotle's earlier ideas about lightness and heaviness of objects, and determined that the interactions between bodies were due to their respective masses. [2, p. 281] The planets interacted with the sun, and formed elliptical orbits.

In the *Principia*, Newton was able to unite the theories of Copernicus, Galileo, and Kepler, and fill in some of the gaps in their models. Newton's Laws of Motion became the widely accepted rules for motion in the universe. Newton set forth new, clear definitions of mass, momentum, inertia, and forces. His three laws of motion consist of: the law of inertia, that an object in motion will remain in motion and an object at rest will remain at rest; the law that acceleration is proportional to force exerted on an object; and the principle of action and reaction, that when an object is acted on by a force, it will have an equal and opposite reaction. Gravity was postulated as the universal force, which obeyed the inverse-square law. The force of gravity between objects was believed to be inversely proportional to the square of the distance between two objects.

Newton's theory required simplifying assumptions and approximations. It was discovered that there were many difficulties applying the laws that worked on earth to the motion of heavenly bodies. There was limited agreement between experimental calculations and observations of the celestial bodies. [3, p. 32] Thinkers following Newton endeavored to derive the mathematics and mechanical theory necessary to create another such theory that was more coherent and universal in its applications. [3, p. 33]

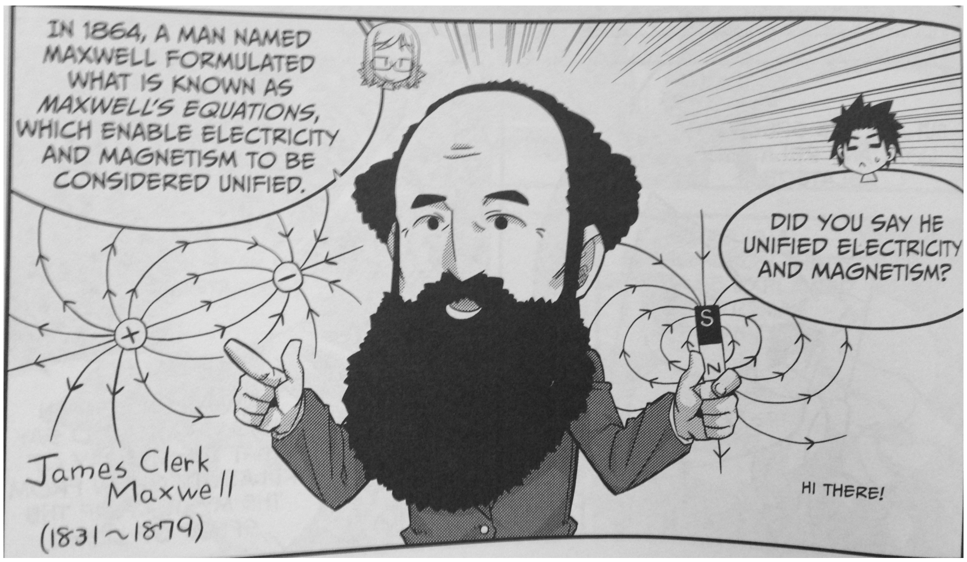

Figure 1: Maxwell, [4, p. 24]

In 1865, Maxwell's mathematical derivations showed that magnets and electrical currents should be able to produce waves capable of moving through space free of the magnets and wires that produced them. [5, p. 127] Maxwell's calculations required that the waves travel at 299,793 kilometers per second, which is the same speed at which light travels. This speed was first measured in the seventeenth century and was a well-known and accepted quantity. Nothing about light was used in Maxwell's derivation, so his discovery resulted in a proposal that light was a traveling wave of electromagnetic energy. [5, p. 129]

Just like water waves and sound waves require a medium through which to propagate, water and air respectively, it was believed that electromagnetic waves required a medium as well. This medium was called the "ether", and it made up all of space, since light could move through anywhere.[5, p. 131] Michelson and Morley set out to determine the speed of the earth by measuring the relative speeds of electromagnetic waves traveling in different directions, but their experiment resulting in not being able to measure the earth's speed through the ether. Einstein would propose that this was because the ether did not exist. Light and electromagnetic waves travel through a vacuum. [5, p. 134-135]

2 Einstein's Theory

2.1 Special Relativity

It was shown that neither electromagnetic nor mechanical phenomena could determine a state of absolute rest, and so Einstein reasoned that such a state did not exist. In short, his theory of relativity states that all observers moving at constant speeds, in inertial frames of reference, will witness the same laws of physics. Someone standing on earth and someone in a rocket accelerating at 9.8 m/s will witness the same laws of physics because the forces acting on their frame of reference are the same. In addition, Einstein postulated that the speed of light is identical for all observers. Experiments showed that light always travelled at the same speed, no matter the speed of the observer. [5, p. 136]

Einstein's theory of relativity states that motion is relative to the observer (fig 2) and that an observer in an inertial reference frame cannot, through experimentation, determine their speed (fig 3). Since the speed of light is always constant, speeds cannot add as in Newton's mechanics. Since there is a "universal speed limit," the speed of light, if something is moving at or near the speed of light, we cannot perform the following calculation: $u_{ground} = c + v$, where u is the speed measured by an observer, and a ball is thrown at the speed of light, c, on a train traveling at speed v. Instead, the correct measurement is: $u_{ground} = \dfrac{u_{train} + v}{1 + vu_{train}/c^2}$, and the "speed limit" is taken into account. [5, p. 140]

2.2 General Relativity

In the general theory, Einstein allows uniform acceleration of reference frames, and takes gravity into account. [6, p. 114] The curvature of space is, in fact, due to energy and gravitation. Figure 4 illustrates an extension of the special theory, as shown in figure 3. Figure 3 illustrated that one could not test their speed using physics; Figure 4 shows that at constant acceleration, in a constant gravitational field, one cannot use physics to tell the difference between the fields, if they are at the same acceleration.

Einstein's field equations describe the geometry of space-time, which depends on the amount of gravitating matter in the region under investigation. [8, p. 112] His field equation is [8, p. 113]:

$$G^{\mu\nu} = R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu} = -8\pi G/c^4 T^{\mu\nu}$$

Where,

$G^{\mu\nu}$ is the Einstein tensor

$R^{\mu\nu}$ is the Ricci curvature tensor

R is the scalar curvature

$g^{\mu\nu}$ is the metric tensor

G is the gravitational constant

c is the speed of light

$T^{\mu\nu}$ is the stress-energy tensor

The Einstein field equations relates a set of 4×4 tensors, each with 10 individual components. They are a system of 10 coupled, nonlinear, partial differential equations, and thus very difficult to solve.

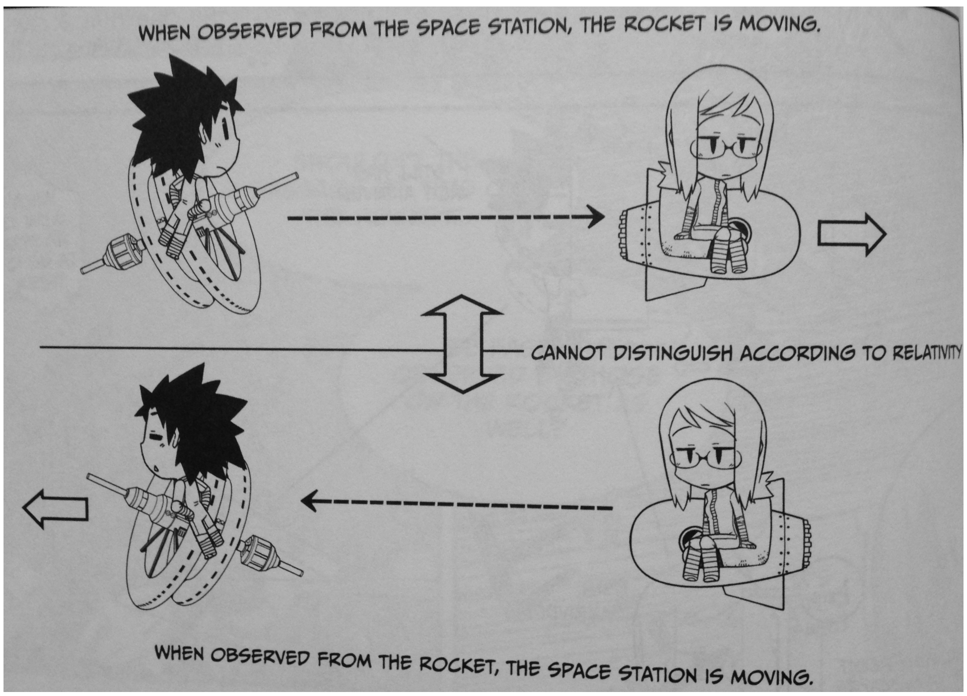

Figure 2: Motion is relative to the observer [4, p. 66]

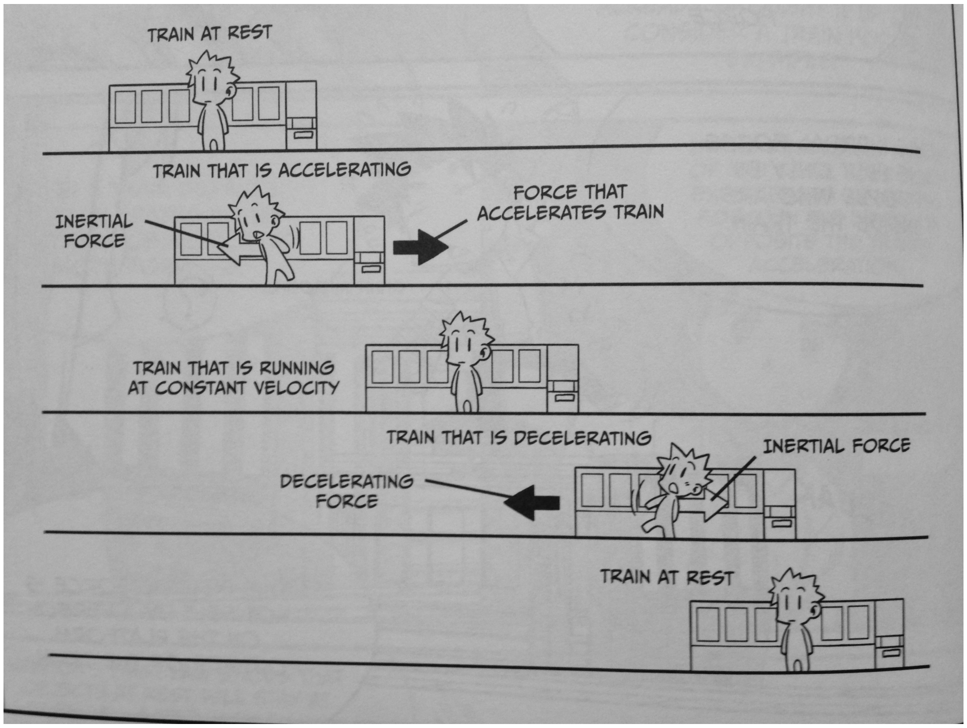

Figure 3: Cannot tell difference between train at rest and at constant velocity [4, p. 124]

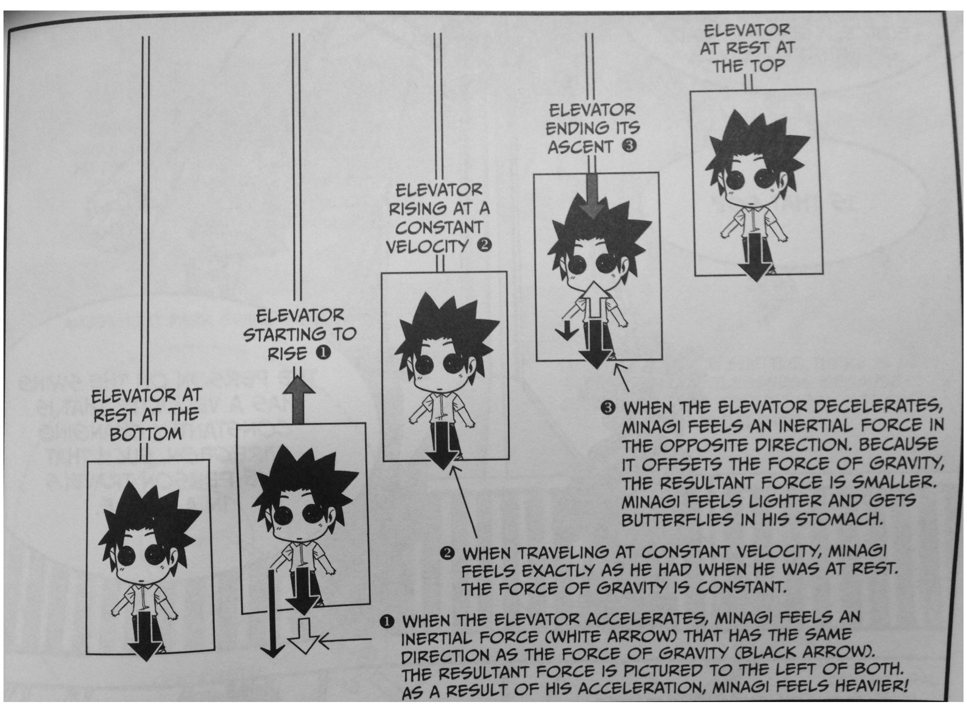

Figure 4: Cannot tell difference between elevator at rest and at constant velocity [4, p. 127]

2.3 Results of general relativity

Mercury's path around the sun differed very greatly and notably from that predicted by Newton's theory. Previously, it was believed that Mercury's shift in perihelion may have been due to the gravitational pull of an extra planet between the sun and Mercury. However, Einstein's theory showed that the orbit of the planets would shift slightly in the sun's gravitational field. Mercury's perihelion shift was most noticeable due to its proximity to the sun.

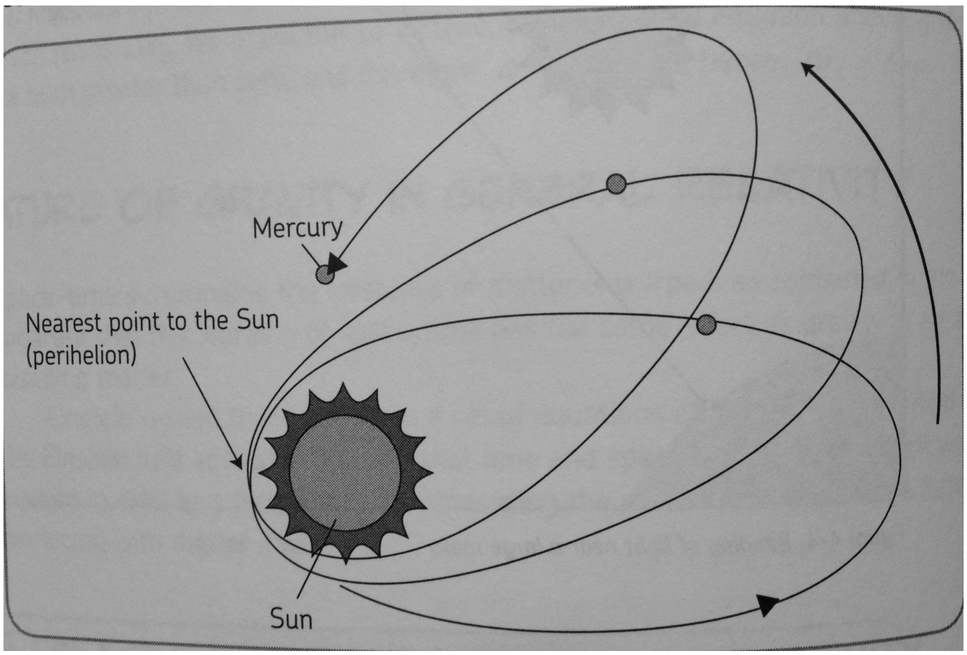

Figure 5: Anomalous shift in perihelion of Mercury [4, p. 164]

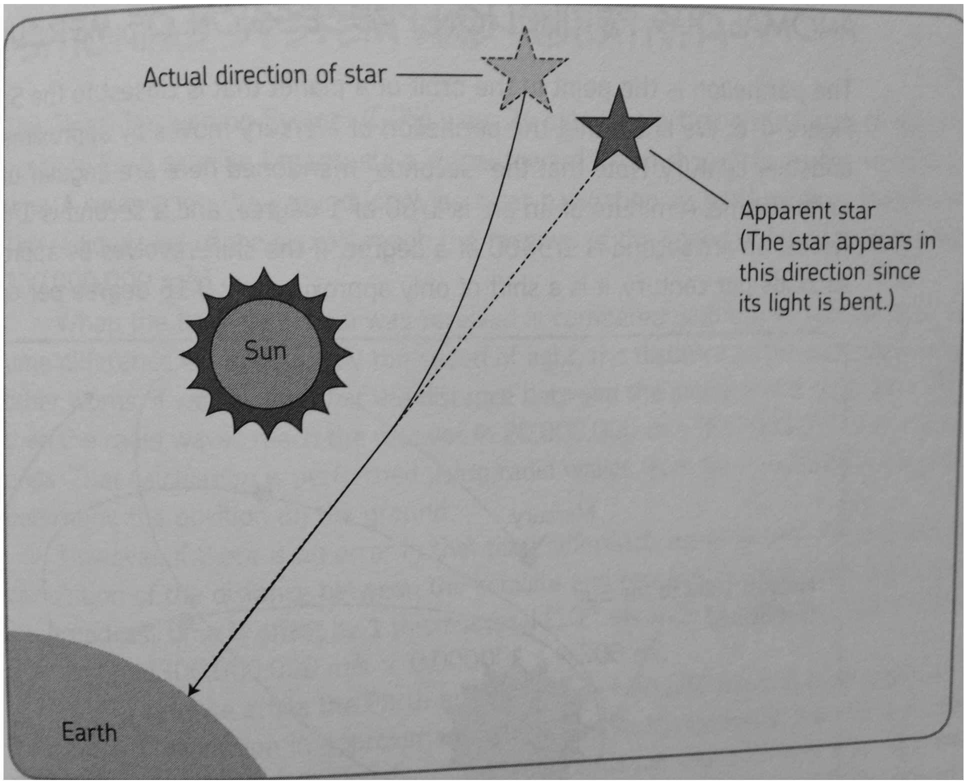

Figure 6: Bending of light near a large mass [4, p. 163]

Though we may observe stars or planets at a given location, due to the effect of the bending of light near massive objects, stars are actually located in other directions.

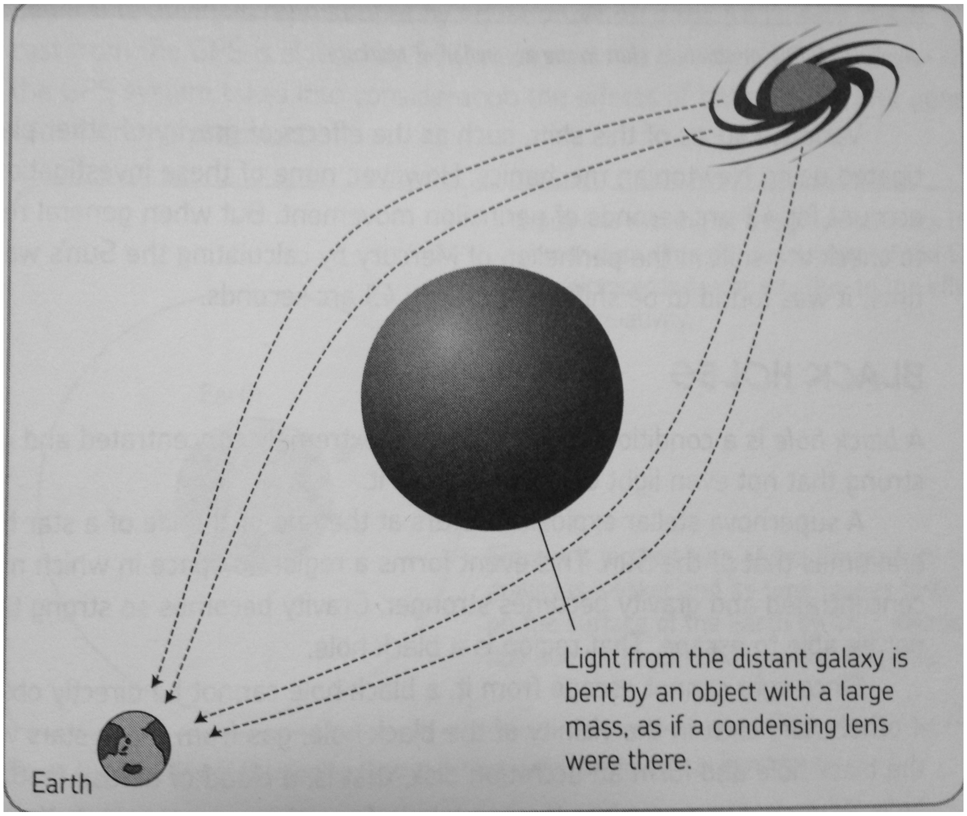

Figure 7: Gravitational lensing [4, p. 163]

Our use of GPS today depends on calculations that derive from general relativity. The GPS satellites are far from the surface of the earth so their experience of time is slightly different. Thus, they have to be calibrated to agree with clocks on the earth.

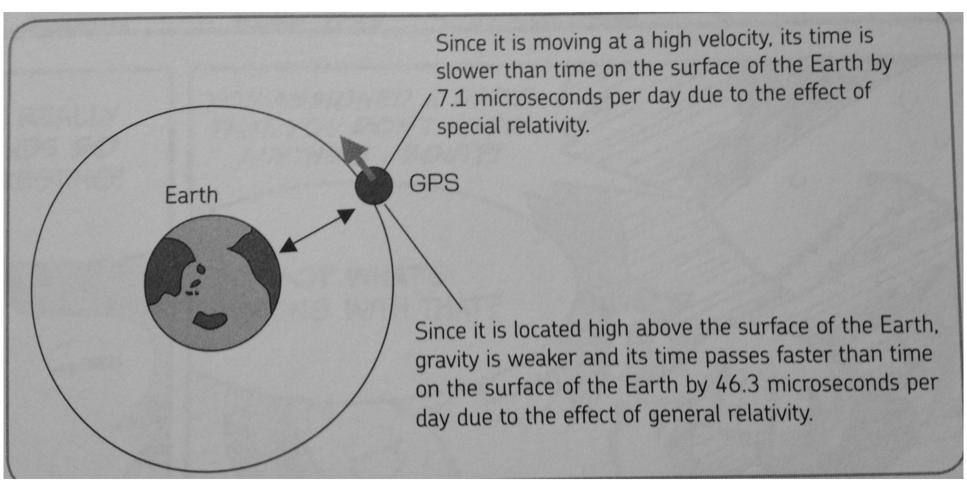

Figure 8: GPS compensates for the effects of general relativity [4, p. 165]

3 Solving for the Schwarzschild Solution

There is no general solution to Einstein's equations, so we must investigate specific cases, in which there is assumed symmetry, and other simplifying assumptions, such as about distribution of matter. [7, p. 137] The first exact solution was obtained by Karl Schwarzschild in 1916. He assumed that the metric tensor field was that for a static, spherically symmetric gravitational field in the empty space-time surrounding a massive spherical object. [8, p. 117] Schwarzschild assumed that: the field was static; that field was spherically symmetric; the space-time was empty; the space-time was asymptotically flat. [8, p. 117] He coordinated by (r, t, θ, φ), and proposed the metric, which is described below. [8, p. 117]

Applications of the Schwarzschild solution include perihelion advance, the bending of light, time delay in radar sounding, and the geodesic effect. [8, p. 123] We can also apply the Schwarzschild solution to a black hole, in which we have a Schwarzschild radius of a massive body. [8, p. 152] If a star, or test particle, is near enough to a dense star that has a Schwarzschild radius, the star will lose mass, which will flow into the dense star. [7, p. 196] Since the system is rotating, the material will approach the black hole in a spiralic motion. The matter will lose energy as it is pulled closer and closer to the Schwarzschild radius, until it is absorbed into the black hole. [7, p. 196]

3.1 The Schwarzschild metric

$$c^2 d\tau^2 = \left(1 - \frac{r_s}{r}\right) c^2 dt^2 - \left(1 - \frac{r_s}{r}\right)^{-1} dr^2 - r^2(d\theta^2 + \sin^2\theta d\varphi^2)$$

c^2 : speed of light

τ : proper time, time measured by a clock moving along the world line of the test particle

r_s : Schwarzschild radius of the massive body

r : the radial coordinate

t : time coordinate, measured by a stationary clock infinitely far from the massive body

θ : the colatitude

φ : the longitude

3.2 The metric tensor matrix

$$\begin{pmatrix} g_{11} & g_{12} & g_{13} & g_{14} \\ g_{21} & g_{22} & g_{23} & g_{24} \\ g_{31} & g_{32} & g_{33} & g_{34} \\ g_{41} & g_{42} & g_{43} & g_{44} \end{pmatrix}$$

The metric tensor matrix is used to describe the metric of the system, and is used to describe the geometry of the space.

$$g_{ij}[\mathbf{f}] = g\left(\frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j}\right)$$

$$\mathbf{f} = \left(\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2}, \frac{\partial}{\partial x^3}, \frac{\partial}{\partial x^4}\right) = \left(\frac{\partial}{\partial t}, \frac{\partial}{\partial r}, \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \varphi}\right)$$

Using the Schwarzschild metric, we arrive at:

$$g\left(\frac{\partial}{\partial t}, \frac{\partial}{\partial t}\right) = \left(1 - \frac{r_s}{r}\right) c^2$$

$$g\left(\frac{\partial}{\partial r}, \frac{\partial}{\partial r}\right) = -\left(1 - \frac{r_s}{r}\right)^{-1}$$

$$g\left(\frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta}\right) = -r^2$$

$$g\left(\frac{\partial}{\partial \varphi}, \frac{\partial}{\partial \varphi}\right) = -r^2 \sin^2 \theta$$

And all other values for g_{ij} = zero.

Now, the g_{ij} matrix =

$$\begin{pmatrix} \left(1 - \frac{r_s}{r}\right) c^2 & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{r_s}{r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}$$

And the inverse matrix, $g^{im} =$

$$\begin{pmatrix} \left(\left(1-\frac{r_s}{r}\right)c^2\right)^{-1} & 0 & 0 & 0 \\ 0 & -\left(1-\frac{r_s}{r}\right) & 0 & 0 \\ 0 & 0 & -r^{-2} & 0 \\ 0 & 0 & 0 & \left(-r^2\sin^2\theta\right)^{-1} \end{pmatrix}$$

3.3 Christoffel symbols

$$0 = \Gamma^i_{kl} = \frac{1}{2} g^{im} \left(\frac{\partial g_{mk}}{\partial x^l} + \frac{\partial g_{ml}}{\partial x^k} - \frac{\partial g_{kl}}{\partial x^m} \right)$$

example calculation:

Choose i = 1, and run through k,l,m =1,2,3,4

$$\Gamma^1_{11} = \frac{1}{2} g^{1m} \left(\frac{\partial g_{m1}}{\partial t} + \frac{\partial g_{m1}}{\partial t} - \frac{\partial g_{11}}{\partial x^m} \right) = 0$$

$$m = 1 : \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial t} + \frac{\partial g_{11}}{\partial t} - \frac{\partial g_{11}}{\partial t} \right) = \frac{1}{2} \left(1 - \frac{r_s}{r} \right)^{-1} \left(\frac{\partial \left(1 - \frac{r_s}{r} \right) c^2}{\partial t} + \frac{\partial \left(1 - \frac{r_s}{r} \right) c^2}{\partial t} - \frac{\partial \left(1 - \frac{r_s}{r} \right) c^2}{\partial t} \right)$$

$$m = 2 : \frac{1}{2} g^{12} \left(\frac{\partial g_{21}}{\partial t} + \frac{\partial g_{21}}{\partial t} - \frac{\partial g_{11}}{\partial r} \right) = \frac{1}{2} (0) \left(0 + 0 - \frac{\partial \left(1 - \frac{r_s}{r} \right) c^2}{\partial r} \right) = 0$$

$$m = 3 : \frac{1}{2} g^{13} \left(\frac{\partial g_{31}}{\partial t} + \frac{\partial g_{13}}{\partial t} - \frac{\partial g_{11}}{\partial \theta} \right) = \frac{1}{2} (0) \left(0 + 0 - \frac{\partial \left(1 - \frac{r_s}{r} \right) c^2}{\partial \theta} \right) = 0$$

$$m = 4 : \frac{1}{2} g^{14} \left(\frac{\partial g_{41}}{\partial t} + \frac{\partial g_{14}}{\partial t} - \frac{\partial g_{11}}{\partial \varphi} \right) = \frac{1}{2} (0) \left(0 + 0 - \frac{\partial \left(1 - \frac{r_s}{r} \right) c^2}{\partial \varphi} \right) = 0$$

$$\Gamma^1_{21} = \Gamma^1_{12} = \frac{1}{2} g^{1m} \left(\frac{\partial g_{m1}}{\partial t} + \frac{\partial g_{m1}}{\partial r} - \frac{\partial g_{21}}{\partial x^m} \right) = \frac{\frac{r_s}{r}}{2\left(1 - \frac{r_s}{r}\right)}$$

$$m = 1: \frac{1}{2} g^{11} \left(\frac{\partial g_{12}}{\partial t} + \frac{\partial g_{11}}{\partial r} - \frac{\partial g_{21}}{\partial t} \right) = \frac{1}{2} \left(1 - \frac{r_s}{r}\right)^{-1} \left(0 + \frac{r_s}{r^2} c^2 - 0\right) = \frac{\frac{r_s}{r} c^2}{2\left(1 - \frac{r_s}{r}\right) c^2}$$

$$m = 2: \frac{1}{2} g^{12} \left(\frac{\partial g_{22}}{\partial t} + \frac{\partial g_{21}}{\partial r} - \frac{\partial g_{21}}{\partial r} \right) = 0$$

$$m = 3: \frac{1}{2} g^{13} \left(\frac{\partial g_{32}}{\partial t} + \frac{\partial g_{23}}{\partial r} - \frac{\partial g_{21}}{\partial \theta} \right) = 0$$

$$m = 4: \frac{1}{2} g^{14} \left(\frac{\partial g_{42}}{\partial t} + \frac{\partial g_{24}}{\partial r} - \frac{\partial g_{21}}{\partial \varphi} \right) = 0$$

$$\Gamma^1_{13} = \Gamma^1_{31} = \frac{1}{2} g^{1m} \left(\frac{\partial g_{m1}}{\partial \theta} + \frac{\partial g_{m1}}{\partial t} - \frac{\partial g_{13}}{\partial x^m} \right) = 0$$

$$m = 1: \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial \theta} + \frac{\partial g_{11}}{\partial t} - \frac{\partial g_{13}}{\partial t} \right) = 0$$

$$m = 2: \frac{1}{2} g^{12} \left(\frac{\partial g_{21}}{\partial \theta} + \frac{\partial g_{21}}{\partial t} - \frac{\partial g_{13}}{\partial r} \right) = 0$$

$$m = 3: \frac{1}{2} g^{11} \left(\frac{\partial g_{31}}{\partial \theta} + \frac{\partial g_{13}}{\partial t} - \frac{\partial g_{13}}{\partial \theta} \right) = 0$$

$$m = 4: \frac{1}{2} g^{14} \left(\frac{\partial g_{41}}{\partial \theta} + \frac{\partial g_{14}}{\partial t} - \frac{\partial g_{13}}{\partial \varphi} \right) = 0$$

$$\Gamma^1_{14} = \Gamma^1_{41} = \frac{1}{2} g^{1m} \left(\frac{\partial g_{m1}}{\partial \varphi} + \frac{\partial g_{m4}}{\partial t} - \frac{\partial g_{41}}{\partial x^m} \right) = 0$$

$$m = 1 : \frac{1}{2} g^{11} \left(\frac{\partial g_{11}}{\partial \varphi} + \frac{\partial g_{14}}{\partial t} - \frac{\partial g_{41}}{\partial t} \right) = 0$$

$$m = 2 : \frac{1}{2} g^{12} \left(\frac{\partial g_{21}}{\partial \varphi} + \frac{\partial g_{24}}{\partial t} - \frac{\partial g_{41}}{\partial r} \right) = 0$$

$$m = 3 : \frac{1}{2} g^{13} \left(\frac{\partial g_{31}}{\partial \varphi} + \frac{\partial g_{34}}{\partial t} - \frac{\partial g_{41}}{\partial \theta} \right) = 0$$

$$m = 4 : \frac{1}{2} g^{14} \left(\frac{\partial g_{41}}{\partial \varphi} + \frac{\partial g_{44}}{\partial t} - \frac{\partial g_{41}}{\partial \varphi} \right) = 0$$

$$\Gamma^1_{22} = \frac{1}{2} g^{1m} \left(\frac{\partial g_{m2}}{\partial r} + \frac{\partial g_{m2}}{\partial r} - \frac{\partial g_{22}}{\partial x^m} \right) = 0$$

$$m = 1 : \frac{1}{2} g^{11} \left(\frac{\partial g_{12}}{\partial r} + \frac{\partial g_{12}}{\partial r} - \frac{\partial g_{22}}{\partial t} \right) = 0$$

$$m = 2 : \frac{1}{2} g^{12} \left(\frac{\partial g_{22}}{\partial r} + \frac{\partial g_{22}}{\partial r} - \frac{\partial g_{22}}{\partial r} \right) = 0$$

$$m = 3 : \frac{1}{2} g^{13} \left(\frac{\partial g_{32}}{\partial r} + \frac{\partial g_{32}}{\partial r} - \frac{\partial g_{22}}{\partial \theta} \right) = 0$$

$$m = 4 : \frac{1}{2} g^{14} \left(\frac{\partial g_{42}}{\partial r} + \frac{\partial g_{42}}{\partial r} - \frac{\partial g_{22}}{\partial \varphi} \right) = 0$$

$$\Gamma^1_{23} = \Gamma^1_{32} = \frac{1}{2} g^{1m} \left(\frac{\partial g_{m2}}{\partial \theta} + \frac{\partial g_{m3}}{\partial r} - \frac{\partial g_{23}}{\partial x^m} \right) = 0$$

$$m = 1 : \frac{1}{2} g^{11} \left(\frac{\partial g_{12}}{\partial \theta} + \frac{\partial g_{13}}{\partial r} - \frac{\partial g_{23}}{\partial t} \right) = 0$$

$$m = 2 : \frac{1}{2} g^{12} \left(\frac{\partial g_{22}}{\partial \theta} + \frac{\partial g_{23}}{\partial r} - \frac{\partial g_{23}}{\partial r} \right) = 0$$

$$m = 3 : \frac{1}{2} g^{13} \left(\frac{\partial g_{32}}{\partial \theta} + \frac{\partial g_{33}}{\partial r} - \frac{\partial g_{23}}{\partial \theta} \right) = 0$$

$$m = 4 : \frac{1}{2} g^{14} \left(\frac{\partial g_{42}}{\partial \theta} + \frac{\partial g_{43}}{\partial r} - \frac{\partial g_{23}}{\partial \varphi} \right) = 0$$

$$\Gamma^1_{24} = \Gamma^1_{42} = \frac{1}{2} g^{1m} \left(\frac{\partial g_{m2}}{\partial \varphi} + \frac{\partial g_{m4}}{\partial r} - \frac{\partial g_{24}}{\partial x^m} \right) = 0$$

$$m = 1 : \frac{1}{2} g^{11} \left(\frac{\partial g_{12}}{\partial \varphi} + \frac{\partial g_{14}}{\partial r} - \frac{\partial g_{24}}{\partial t} \right) = 0$$

$$m = 2 : \frac{1}{2} g^{12} \left(\frac{\partial g_{22}}{\partial \varphi} + \frac{\partial g_{24}}{\partial r} - \frac{\partial g_{24}}{\partial r} \right) = 0$$

$$m = 3 : \frac{1}{2} g^{13} \left(\frac{\partial g_{32}}{\partial \varphi} + \frac{\partial g_{34}}{\partial r} - \frac{\partial g_{24}}{\partial \theta} \right) = 0$$

$$m = 4 : \frac{1}{2} g^{14} \left(\frac{\partial g_{42}}{\partial \varphi} + \frac{\partial g_{44}}{\partial r} - \frac{\partial g_{24}}{\partial \varphi} \right) = 0$$

$$\Gamma^1_{33} = \frac{1}{2} g^{1m} \left(\frac{\partial g_{m3}}{\partial \theta} + \frac{\partial g_{m3}}{\partial \theta} - \frac{\partial g_{33}}{\partial x^m} \right) = 0$$

$$m = 1 : \frac{1}{2} g^{11} \left(\frac{\partial g_{13}}{\partial \theta} + \frac{\partial g_{13}}{\partial \theta} - \frac{\partial g_{33}}{\partial t} \right) = 0$$

$$m = 2 : \frac{1}{2} g^{12} \left(\frac{\partial g_{23}}{\partial \theta} + \frac{\partial g_{23}}{\partial \theta} - \frac{\partial g_{33}}{\partial r} \right) = 0$$

$$m = 3 : \frac{1}{2} g^{13} \left(\frac{\partial g_{33}}{\partial \theta} + \frac{\partial g_{33}}{\partial \theta} - \frac{\partial g_{33}}{\partial \theta} \right) = 0$$

$$m = 4 : \frac{1}{2} g^{14} \left(\frac{\partial g_{43}}{\partial \theta} + \frac{\partial g_{43}}{\partial \theta} - \frac{\partial g_{33}}{\partial \varphi} \right) = 0$$

$$\Gamma^1_{34} = \Gamma^1_{43} = \frac{1}{2} g^{1m} \left(\frac{\partial g_{m3}}{\partial \varphi} + \frac{\partial g_{m4}}{\partial \theta} - \frac{\partial g_{34}}{\partial x^m} \right) = 0$$

$$m = 1 : \frac{1}{2} g^{11} \left(\frac{\partial g_{13}}{\partial \varphi} + \frac{\partial g_{14}}{\partial \theta} - \frac{\partial g_{34}}{\partial t} \right) = 0$$

$$m = 2 : \frac{1}{2} g^{12} \left(\frac{\partial g_{23}}{\partial \varphi} + \frac{\partial g_{24}}{\partial \theta} - \frac{\partial g_{34}}{\partial r} \right) = 0$$

$$m = 3 : \frac{1}{2} g^{13} \left(\frac{\partial g_{33}}{\partial \varphi} + \frac{\partial g_{34}}{\partial \theta} - \frac{\partial g_{34}}{\partial \theta} \right) = 0$$

$$m = 4 : \frac{1}{2} g^{14} \left(\frac{\partial g_{43}}{\partial \varphi} + \frac{\partial g_{44}}{\partial \theta} - \frac{\partial g_{34}}{\partial \varphi} \right) = 0$$

$$\Gamma^1_{44} = \frac{1}{2}g^{1m}\left(\frac{\partial g_{m4}}{\partial \varphi} + \frac{\partial g_{m4}}{\partial \varphi} - \frac{\partial g_{44}}{\partial x^m}\right) = 0$$

$$m = 1 : \frac{1}{2}g^{11}\left(\frac{\partial g_{14}}{\partial \varphi} + \frac{\partial g_{14}}{\partial \varphi} - \frac{\partial g_{44}}{\partial t}\right) = 0$$

$$m = 2 : \frac{1}{2}g^{12}\left(\frac{\partial g_{24}}{\partial \varphi} + \frac{\partial g_{24}}{\partial \varphi} - \frac{\partial g_{44}}{\partial r}\right) = 0$$

$$m = 3 : \frac{1}{2}g^{13}\left(\frac{\partial g_{34}}{\partial \varphi} + \frac{\partial g_{34}}{\partial \varphi} - \frac{\partial g_{44}}{\partial \theta}\right) = 0$$

$$m = 4 : \frac{1}{2}g^{14}\left(\frac{\partial g_{44}}{\partial \varphi} + \frac{\partial g_{44}}{\partial \varphi} - \frac{\partial g_{44}}{\partial \varphi}\right) = 0$$

Using the metric tensor above, our Christoffel symbols are:

set i=1, $\Gamma^1_{kl} =$

$$\begin{bmatrix} 0 & \frac{\frac{r_s}{r^2}}{2(1-\frac{r_s}{r})} & 0 & 0 \\ \frac{\frac{r_s}{r^2}}{2(1-\frac{r_s}{r})} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

set i=2, $\Gamma^2_{kl} =$

$$\begin{bmatrix} \frac{\frac{r_s}{r^2}c^2}{2(1-\frac{r_s}{r})} & 0 & 0 & 0 \\ 0 & \frac{-\left(\frac{r_s}{r^2}\right)}{2(1-\frac{r_s}{r})} & 0 & 0 \\ 0 & 0 & -r\left(1-\frac{r_s}{r}\right) & 0 \\ 0 & 0 & 0 & \frac{-\left(1-\frac{r_s}{r}\right)(2r\sin^2\theta)}{2} \end{bmatrix}$$

set i=3, $\Gamma^3_{kl} =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{r} & 0 \\ 0 & \frac{1}{r} & 0 & 0 \\ 0 & 0 & 0 & -\sin\theta\cos\theta \end{bmatrix}$$

set i=4, $\Gamma^4_{kl} =$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{r} \\ 0 & 0 & 0 & \cot\theta \\ 0 & \frac{1}{r} & \cot\theta & 0 \end{bmatrix}$$

3.4 Geodesic equation

The geodesic equation describes lines of motion in our metric. These are like straight lines as we are used to them on Earth. Geodesic lines determine the lines of shortest distance in curved space. [9, p. 74] The following calculation is used to find the geodesics for the Schwarzschild solution.

$$0 = \frac{\partial^2 u^i}{\partial \tau^2} + \Gamma^i_{kl} \frac{\partial u^l}{\partial \tau} \frac{\partial u^k}{\partial \tau}$$

sample calculation:

pick i = 1

$$0 = \frac{\partial^2 u^1}{\partial \tau^2} + 2\Gamma^1_{12} \frac{\partial u^1}{\partial \tau} \frac{\partial u^2}{\partial \tau}$$

$$0 = \frac{\partial^2 t}{\partial \tau^2} + 2 \frac{\frac{r_s}{r^2}}{2(1 - \frac{r_s}{r})} \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau}$$

$$0 = \frac{\partial^2 t}{\partial \tau^2} + \frac{\frac{r_s}{r^2}}{(1 - \frac{r_s}{r})} \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau}$$

This is the geodesic equation for when i = 1. Our sum is just $2\Gamma^1_{12}$ since $\Gamma^1_{12} = \Gamma^1_{21}$ by symmetry of Christoffel symbols of the second kind, and all other values for $\Gamma^1_{kl} = 0$.

The geodesic equations for the Schwarzschild metric:

$$0 = \frac{\partial^2 t}{\partial \tau^2} + \frac{w}{v} \frac{\partial t}{\partial \tau} \frac{\partial r}{\partial \tau}$$

$$0 = \frac{\partial^2 r}{\partial \tau^2} + \frac{wc^2}{2v} \left(\frac{\partial t}{\partial \tau} \right)^2 + \frac{-w}{2v} \left(\frac{\partial r}{\partial \tau} \right)^2 + -v(r \sin^2 \theta) \left(\frac{\partial \varphi}{\partial \tau} \right)^2$$

$$0 = -(\sin \theta \cos \theta) \left(\frac{\partial \varphi}{\partial \tau} \right)^2$$

$$0 = \frac{\partial^2 \varphi}{\partial \tau^2} + 2 \left(\frac{1}{r} \right) \frac{\partial \varphi}{\partial \tau} \frac{\partial r}{\partial \tau}$$

where $w = \frac{r_s}{r^2}$ and $v = \left(1 - \frac{r_s}{r^2} \right)$

These equations can be solved as a system of ordinary differential equations. θ is set to $\frac{\pi}{2}$

in order to simplify the equations and make them solvable. The metric is used to determine initial conditions.

3.5 Initial conditions

In order to solve a system of ODEs, we need initial conditions for our variables. The variables are r, t, φ. Since the equations to be solved include second derivatives, initial conditions are needed for the initial values and first derivatives at $\tau = 0$.

We choose a value for r(0) $= r_0$, in terms of initial distance of the photon from the Schwarzschild radius, and set

t(0) $= 0$ and

$\varphi(0) = 0$.

To solve for $\frac{\partial r}{\partial \tau}$, the equation of a circle, $r^2 = x^2 + y^2$ is used. Differentiating, we have: $2r\frac{\partial r}{\partial \tau} = 2x\frac{\partial x}{\partial \tau} + 2y\frac{\partial y}{\partial \tau}$. But we know that the initial distance, x, is the same as our initial radius, r, and initial y $= 0$, since the initial position is along the x-axis. Thus, $\frac{\partial r}{\partial \tau} = \frac{\partial x}{\partial \tau}$. In terms of coordinates, $\frac{\partial x}{\partial \tau} = v\cos\alpha$, where v = velocity of the photon, α is the angle of initial velocity vector, and $\cos\alpha = \frac{\partial x}{\partial \tau}$. We have

$\frac{\partial r}{\partial \tau} = v\cos\alpha$.

To calculate $\frac{\partial \varphi}{\partial \tau}$, we differentiate the relations $\cos\varphi = \frac{x}{r}$ and $\sin\varphi = \frac{y}{r}$. We have $-\frac{\partial \sin\varphi}{\partial \tau} = \frac{r\frac{\partial x}{\partial \tau} - x\frac{\partial r}{\partial \tau}}{r^2}$ and $\frac{\partial \cos\varphi}{\partial \tau} = \frac{r\frac{\partial y}{\partial \tau} - y\frac{\partial r}{\partial \tau}}{r^2}$ respectively. Using initial values $\varphi = 0, x = r, y = 0$ tells us that the first equation gives no information and the second equation yields:

$\frac{\partial \varphi}{\partial \tau} = \frac{\frac{\partial y}{\partial r}}{r}$.

The final initial value is solved using the Schwarzschild metric. Rearranging and substituting, we get:

$\frac{\partial t}{\partial \tau} = \sqrt{\frac{1}{1 - \frac{r_s}{r}}\left(1 + \frac{(\cos\alpha)^2}{r_s}r + \sin^2\alpha\right)}$.

Our initial conditions are:

$$
\begin{aligned}
r(0) &= r_0 \\
t(0) &= 0 \\
\varphi(0) &= 0 \\
\frac{\partial r}{\partial \tau} &= v \cos \alpha \\
\frac{\partial \varphi}{\partial \tau} &= \frac{\frac{\partial y}{\partial r}}{r} \\
\frac{\partial t}{\partial \tau} &= \sqrt{\frac{1}{1 - \frac{r_s}{r}} \left(1 + \frac{(\cos \alpha)^2}{r_s} r + \sin^2 \alpha \right)}
\end{aligned}
$$

3.6 MATLAB code to solve equations

In order to solve the Schwarzschild solution using the MATLAB ODE solver, all equations must be solved in terms of y. We define the variables:

$$
\begin{aligned}
r &= y_1 \\
t &= y_2 \\
\varphi &= y_3 \\
\frac{\partial r}{\partial \tau} &= dy_1 = y_4 \\
\frac{\partial t}{\partial \tau} &= dy_2 = y_5 \\
\frac{\partial \varphi}{\partial \tau} &= dy_3 = y_6 \\
\frac{\partial^2 r}{\partial \tau^2} &= dy_4 = y_7 \\
\frac{\partial^2 t}{\partial \tau^2} &= dy_5 = y_8 \\
\frac{\partial^2 \varphi}{\partial \tau^2} &= dy_6 = y_9
\end{aligned}
$$

Substituting, the Schwarzschild solution equations are:

$$0 = y_8 + \frac{w}{v} y_5 y_6$$

$$0 = y_7 + \frac{w}{2v} c^2 y_5^2 - v(y_1 \sin^2\theta) y_6^2$$

$$0 = -(\sin\theta\cos\theta) y_6^2$$

$$0 = y_9 + \frac{2}{y_1} y_6 y_4$$

The third equation gives us no information, and we must solve in terms of y's. So the ordinary differential equations to be solved are:

$$y_8 = -\frac{w}{v} y_5 y_6$$

$$y_7 = -\frac{w}{2v} c^2 y_5^2 + v(y_1 \sin^2\theta) y_6^2$$

$$y_9 = -\frac{2}{y_1} y_6 y_4$$

The ODE was solved using MATLAB, plotting the path of a photon, $(\frac{\partial r}{\partial \tau}, \frac{\partial \varphi}{\partial \tau})$ and the Schwarzschild radius. The code and some generated figures follow:

```
function geo
%%geodesic equation
%venusaur
%****set options****
options = odeset('RelTol', ...
    1e-3,'AbsTol',1e-6,'NormControl','on','MaxOrder',5);
 %2GM/c^2 = r_s
 %***make sure to change RS in function below!!***
    %1.27e10; %Schwarzschild radius (m) radius of Sagittarius A* (SMBH)
    %2.95e+3; %Shwarzschild radius (m) of sun
  RS=8.87e-3; %Schwarzschild radius (m) of earth (radius of earth = ...
      6371 m)
```

```
v = 299792458; %speed of photon (light) (m/s);

% ***things to change***
r0 = 700000000000000000000000000000000*RS; %initial radius (m)
a = 120; %initial angle of photon (deg)
time = 1000000000000000000000000; %time of plot

% vector of initial conditions:
y0=[r0 0 0 v*cosd(a) sqrt( (1/(1-(RS/r0))) * (1 + ...
    (cosd(a)^2)/(1-(RS/r0)) ...
    + sind(a)^2)) v*sind(a)/(r0)];
%r0, t0, p0, dr/dT (<dx,dy>), dt/dT, dp/dT

%solve system using 15s
[T,y]= ode15s(@defineODE,[0 time],y0,options);
    r=[y(:,1)];
    t=[y(:,2)];
    p=[y(:,3)];
    dr=[y(:,4)];
    dt=[y(:,5)];
    dp=[y(:,6)];

%plot the solution for r and p as a function of proper time T.
polar(p,r,'b'); %graph of photon path
hold on

    xc = 0;
    yc = 0;
    theta = linspace(0,2*pi);
    x = RS*cos(theta) + xc;
    y = RS*sin(theta) + yc;
    plot(x,y,'r') %graph of schwarzschild radius

set(0,'DefaultAxesFontSize',15)
xlabel('angle')
ylabel('radius');
title('path of photon in presence of massive body');
legend('path of photon','Schwarzschild horizon');
grid on;

hold off

    function dy = defineODE(T,y)
```

```
    dy = zeros(length(y0),1);
        %y1=r %y2=t %y3=phi
        c = 299792458; %speed of light (m/s)
        %1.27e+10; %Schwarzschild radius (m) radius of Sagittarius A* ...
            (SMBH)
        %2.95e+3; %Schwarzschild radius (m) of sun
      RS= 8.87e-3; %Schwarzschild radius (m) of earth

        % theta = pi/2
        w = (RS/(y(1).^2));
        v = (1 - (RS./y(1)));
    dy(1) = y(4);
    dy(2) = y(5);
    dy(3) = y(6);
    dy(4) = (-1.*(w./(2.*v))).*((c).^2).*(y(5)).^2+ ...
        (w./(2.*v)).*(y(4)).^2-(v.*(y(1).*sin(pi/2).^2).*y(6).^2); % = y(7)
    dy(5) = (-1.*(w./v)).*y(5).*y(4); % = y(8)
    dy(6) = (-2/y(1)).*y(6).*y(4); % = y(9)

    end
end
```

3.7 Figures of solution

The following figures were generated by the above MATLAB code, using the Schwarzschild radius of the earth. The caption gives information about initial radius of photon, r0, initial angle, α, and observation time, T.

3.7.1 Non-escape examples

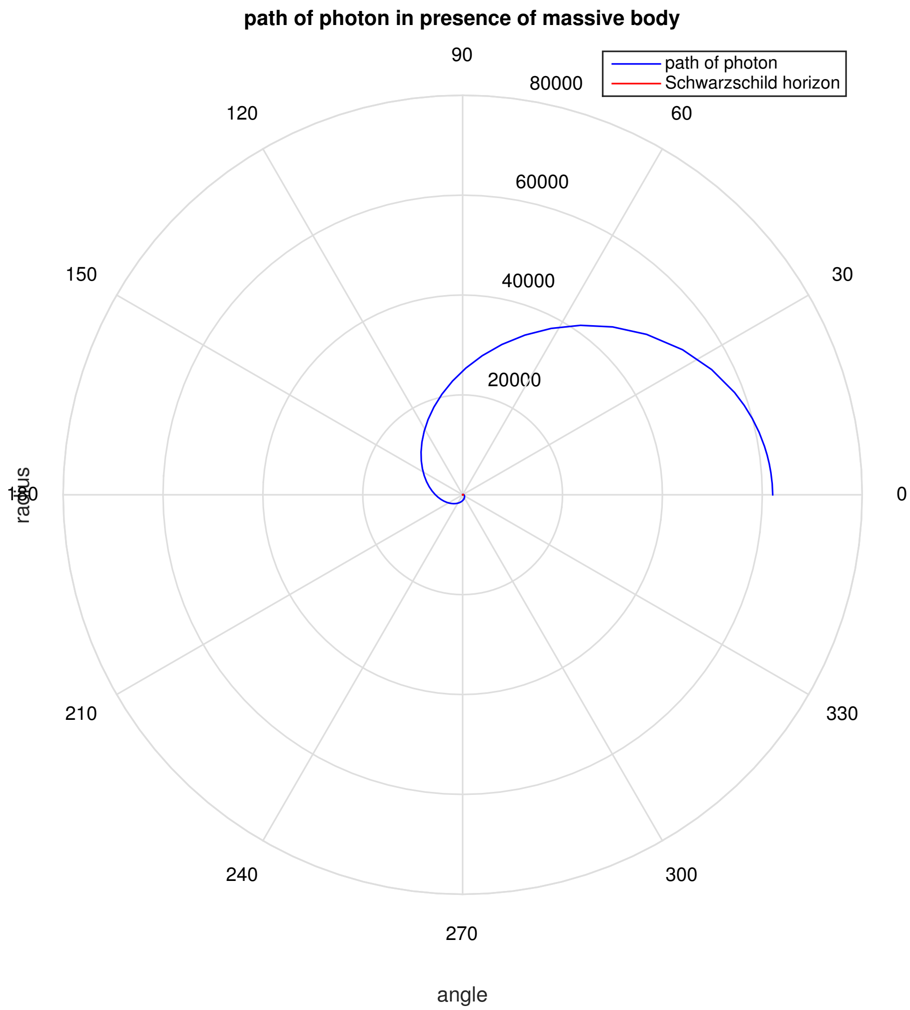

Figure 9: initial radius = 7×10^6 r$_s$, initial angle = 90 deg, running time = 1×10^3

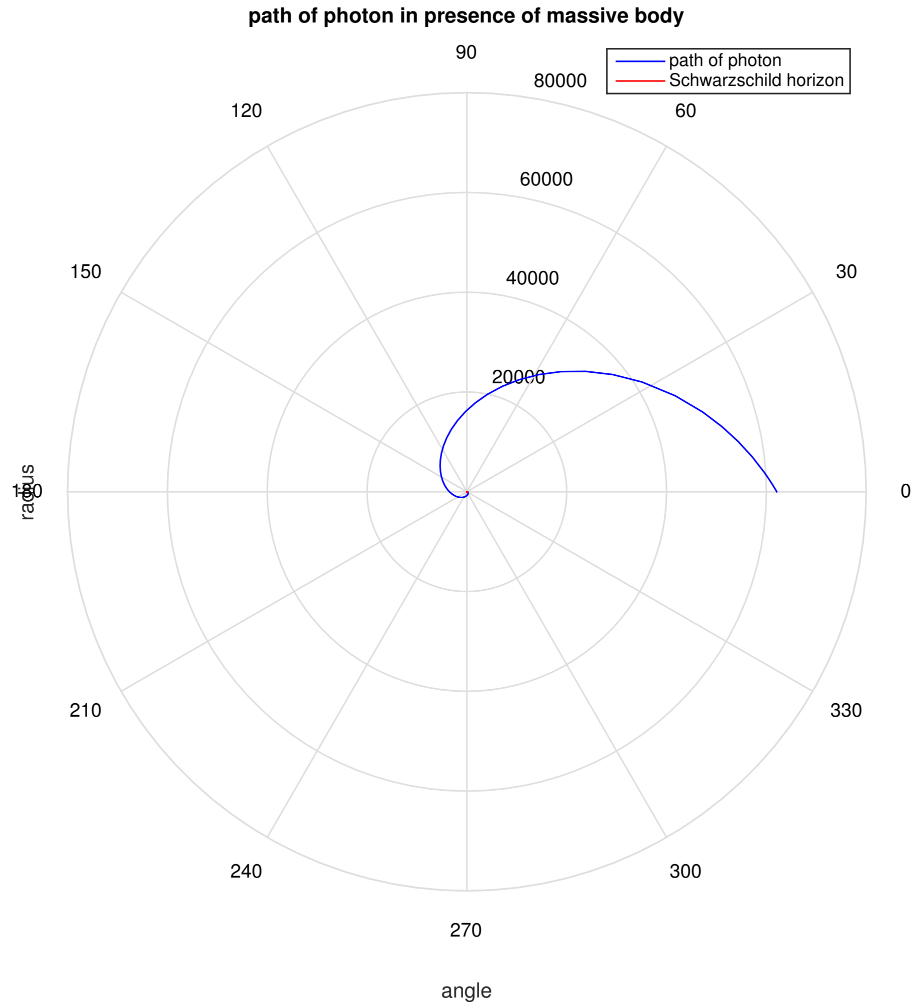

Figure 10: initial radius = $7{\times}10^6$ r$_s$, initial angle = 120 deg, running time = $1{\times}10^3$

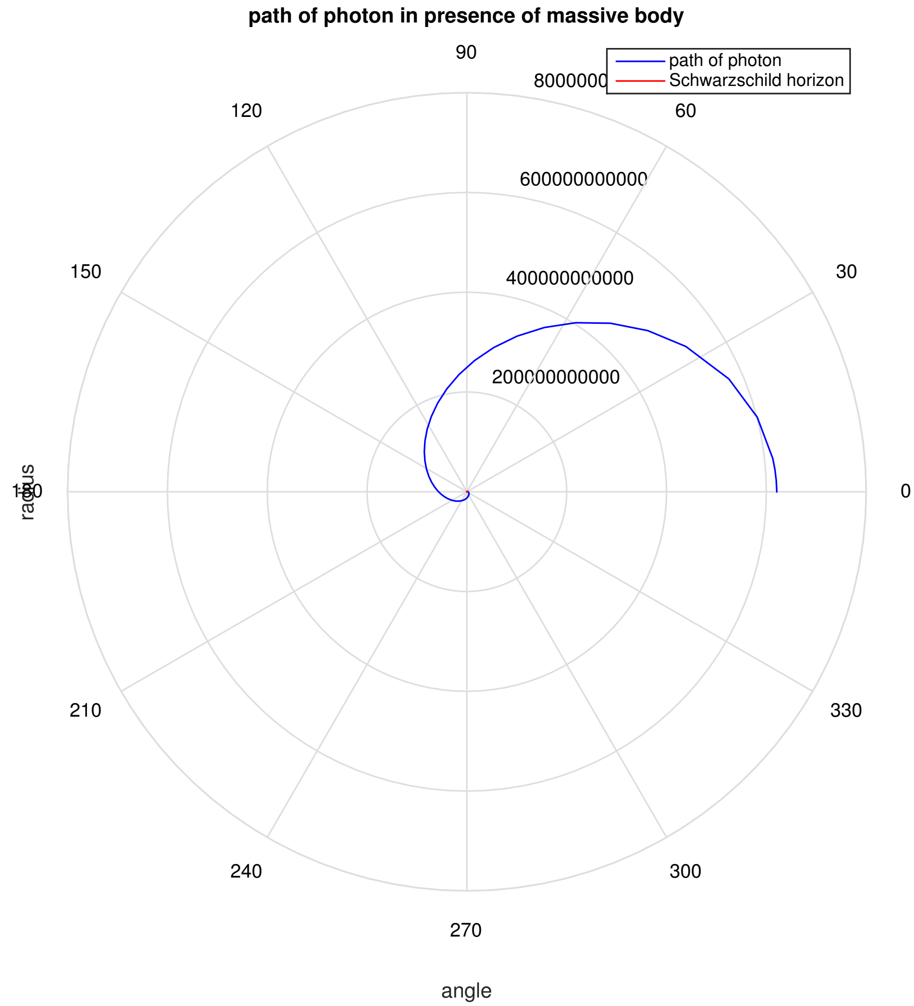

Figure 11: initial radius = $7{\times}10^{13}$ r_s, initial angle = 90 deg, running time = $1{\times}10^4$

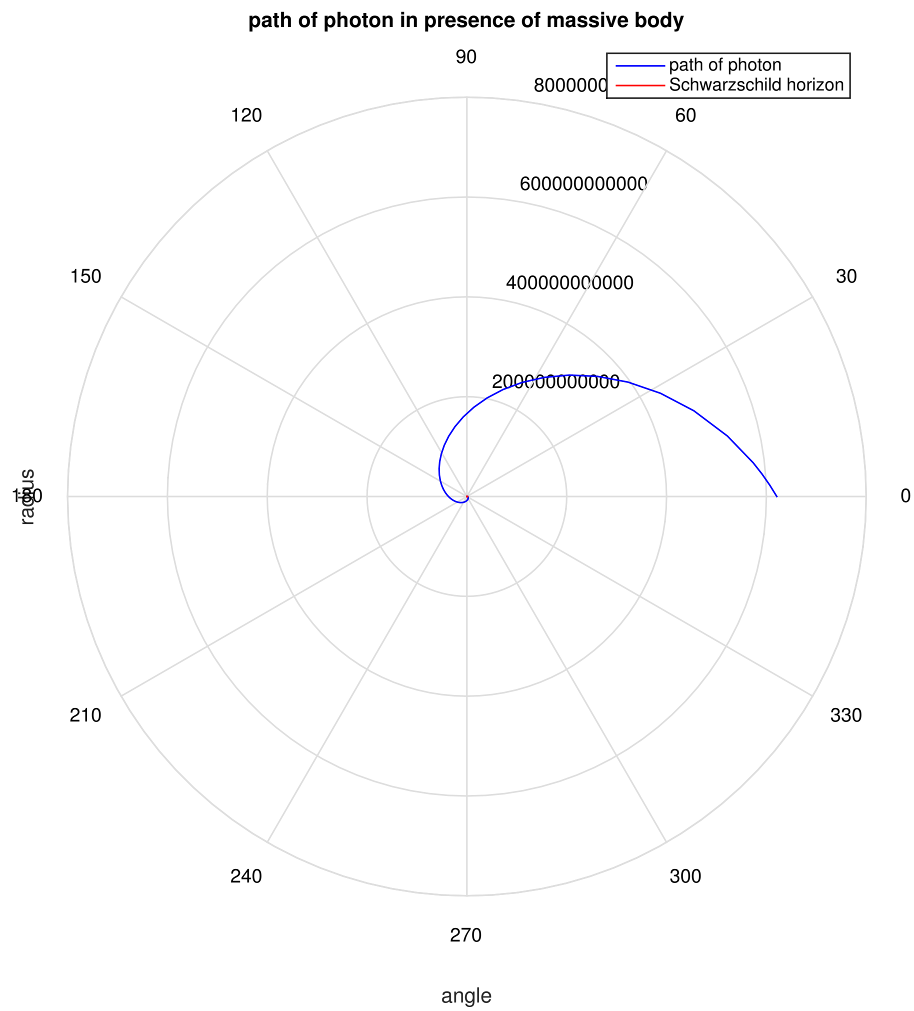

Figure 12: initial radius = $7{\times}10^{13}$ r$_s$, initial angle = 120 deg, running time = $1{\times}10^4$

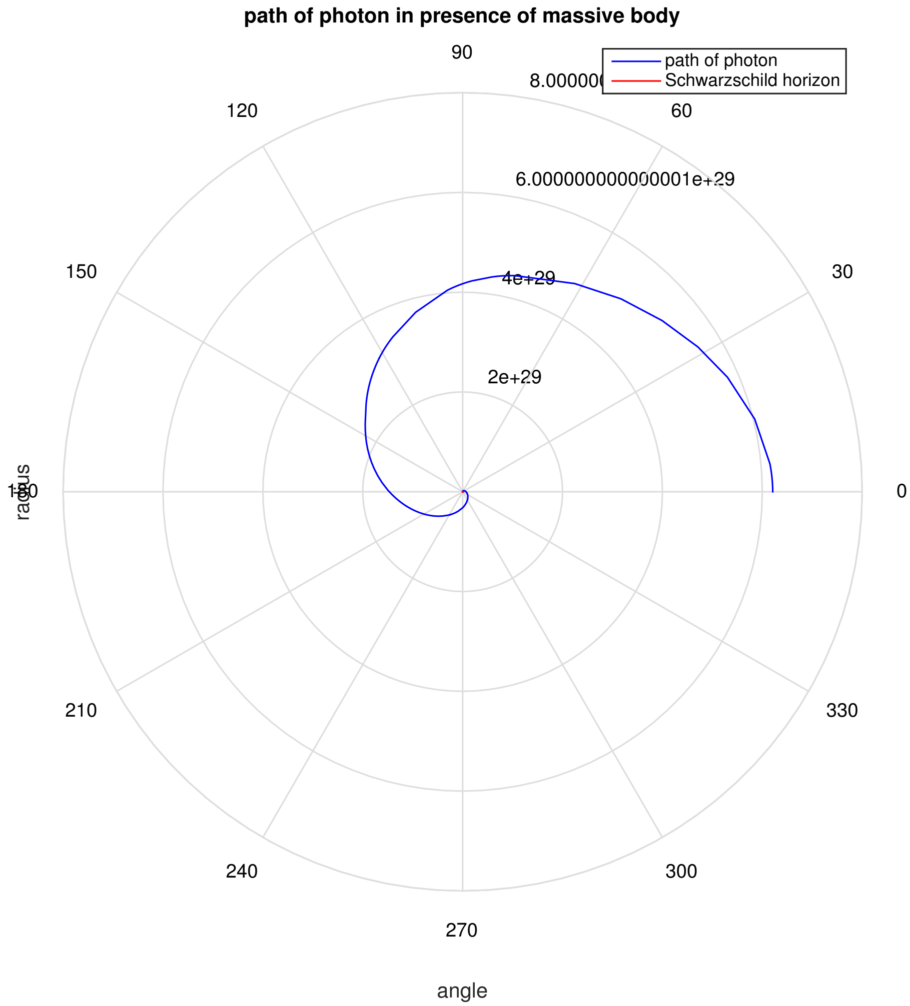

Figure 13: initial radius = $7{\times}10^{31}$ r_s, initial angle = 90 deg, running time = $1{\times}10^{25}$

3.7.2 Escape examples

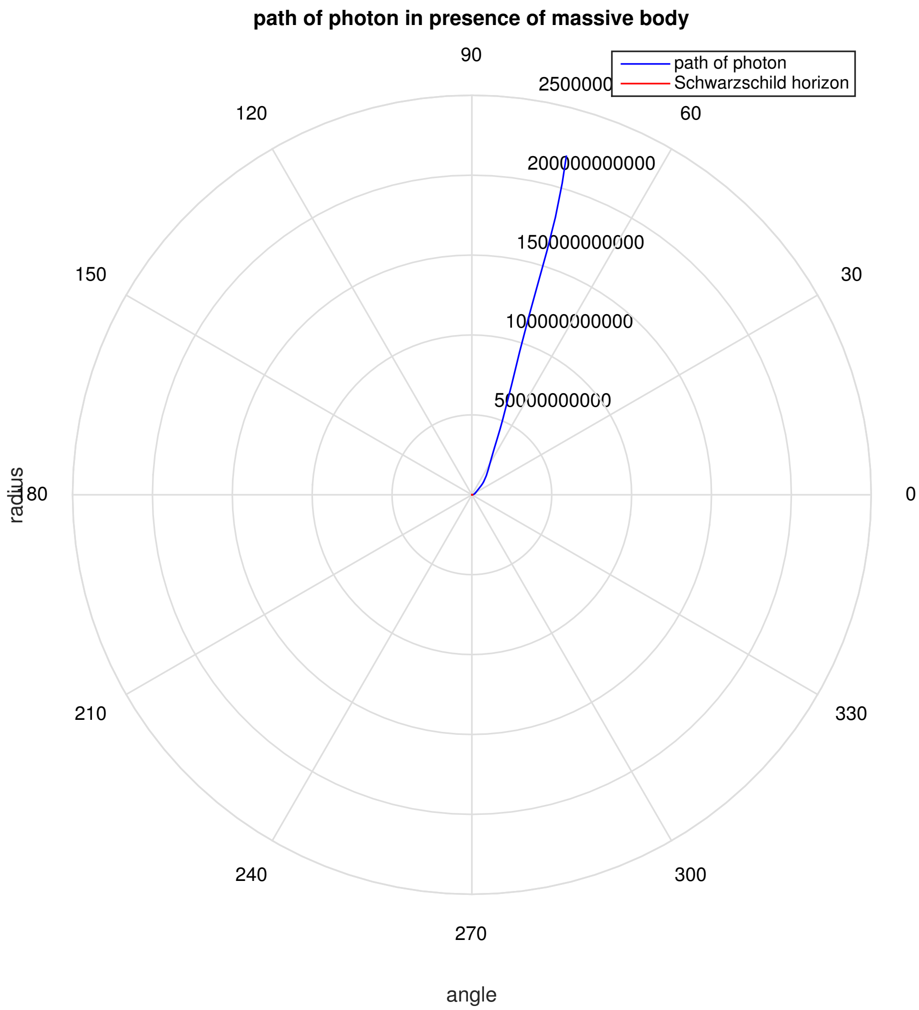

Figure 14: initial radius = 7×10^6r$_s$, initial angle = 10 deg, running time = 1×10^3

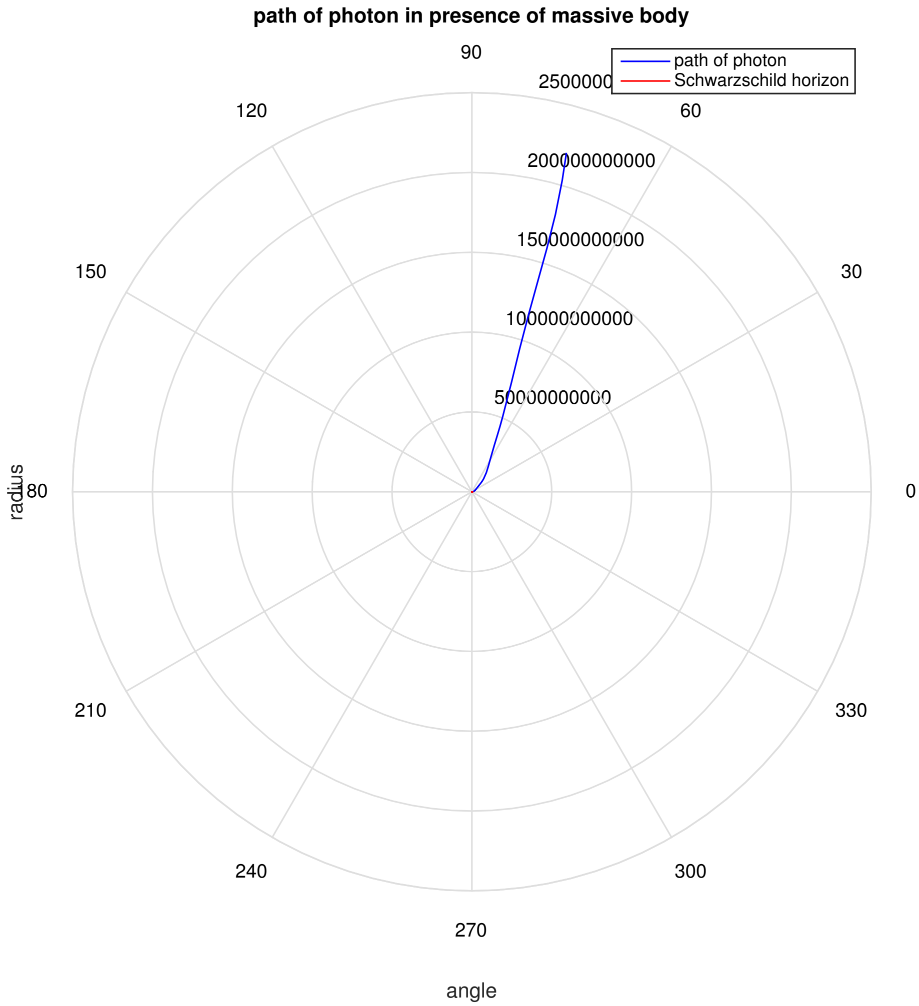

Figure 15: initial radius = $7{\times}10^6 \mathrm{r}_s$, initial angle = 40 deg, running time = $1{\times}10^3$

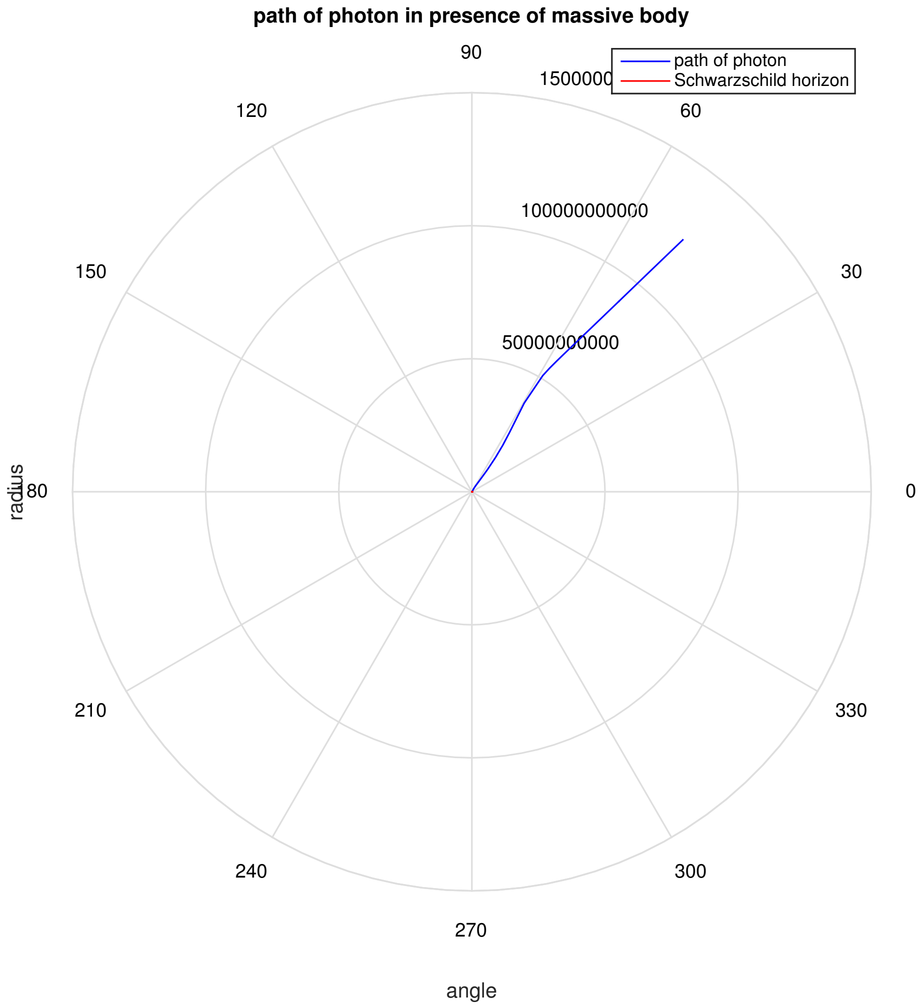

Figure 16: initial radius = $7{\times}10^9$r$_s$, initial angle = 40 deg, running time = $1{\times}10^3$

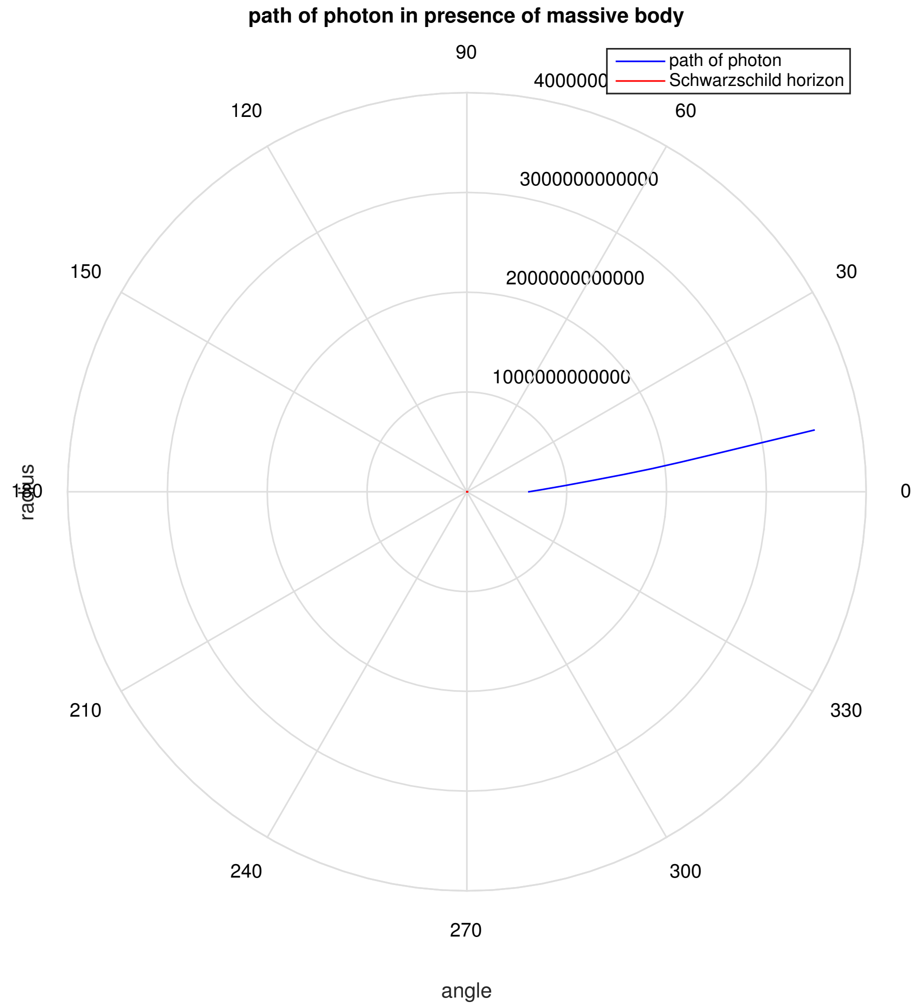

Figure 17: initial radius = $7{\times}10^{13}$r$_s$, initial angle = 10 deg, running time = $1{\times}10^4$

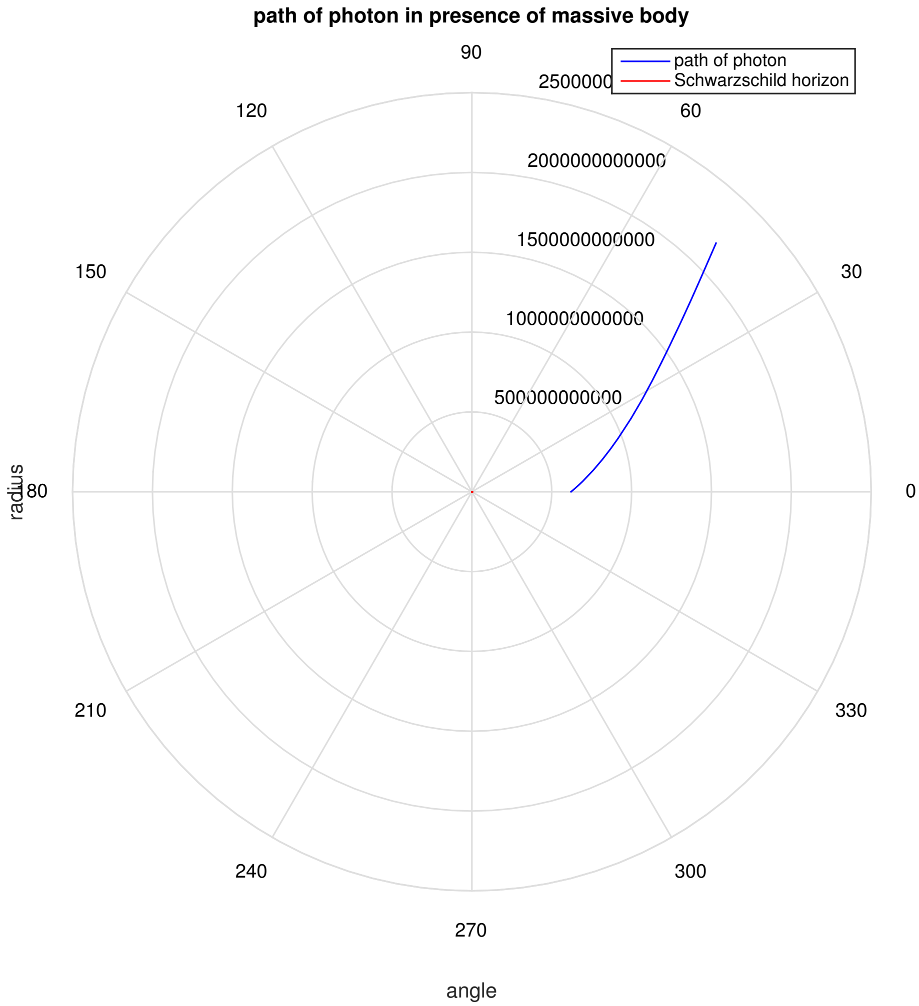

Figure 18: initial radius = 7×10^{13}r$_s$, initial angle = 40 deg, running time = 1×10^4

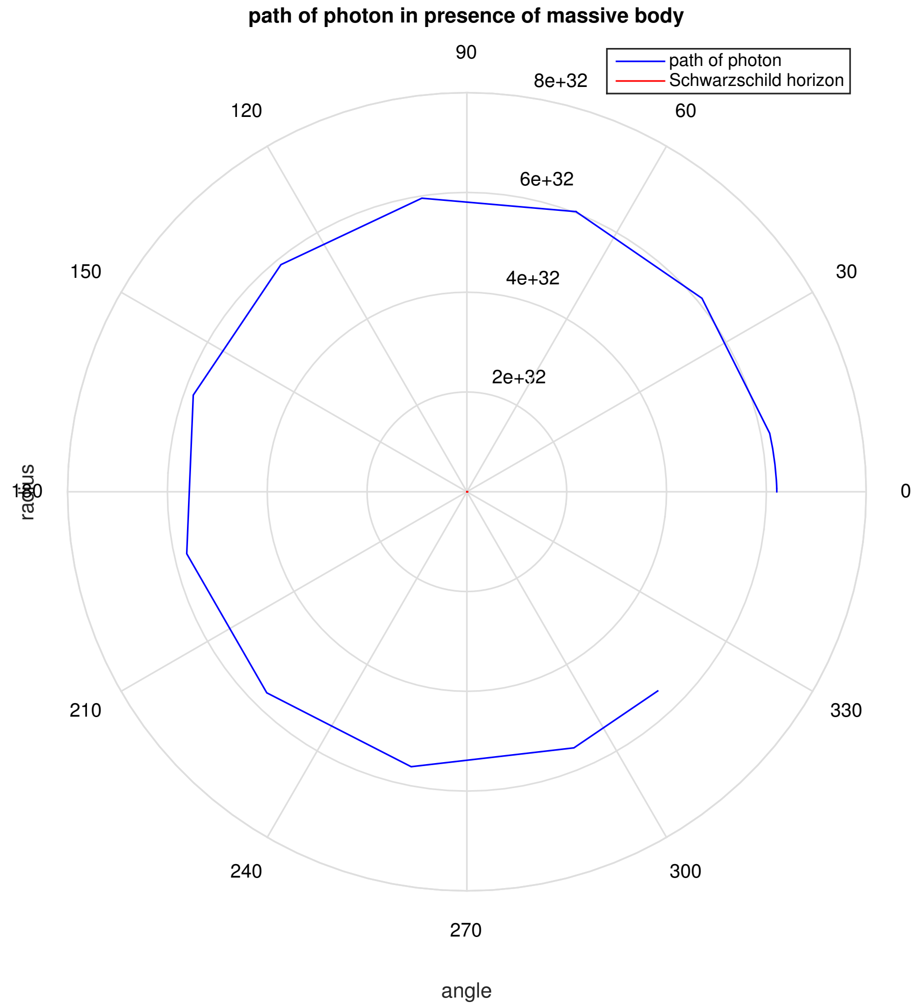

Figure 19: initial radius = 7 $\times 10^{34}$r$_s$, initial angle = 90 deg, running time = 1 $\times 10^{24}$

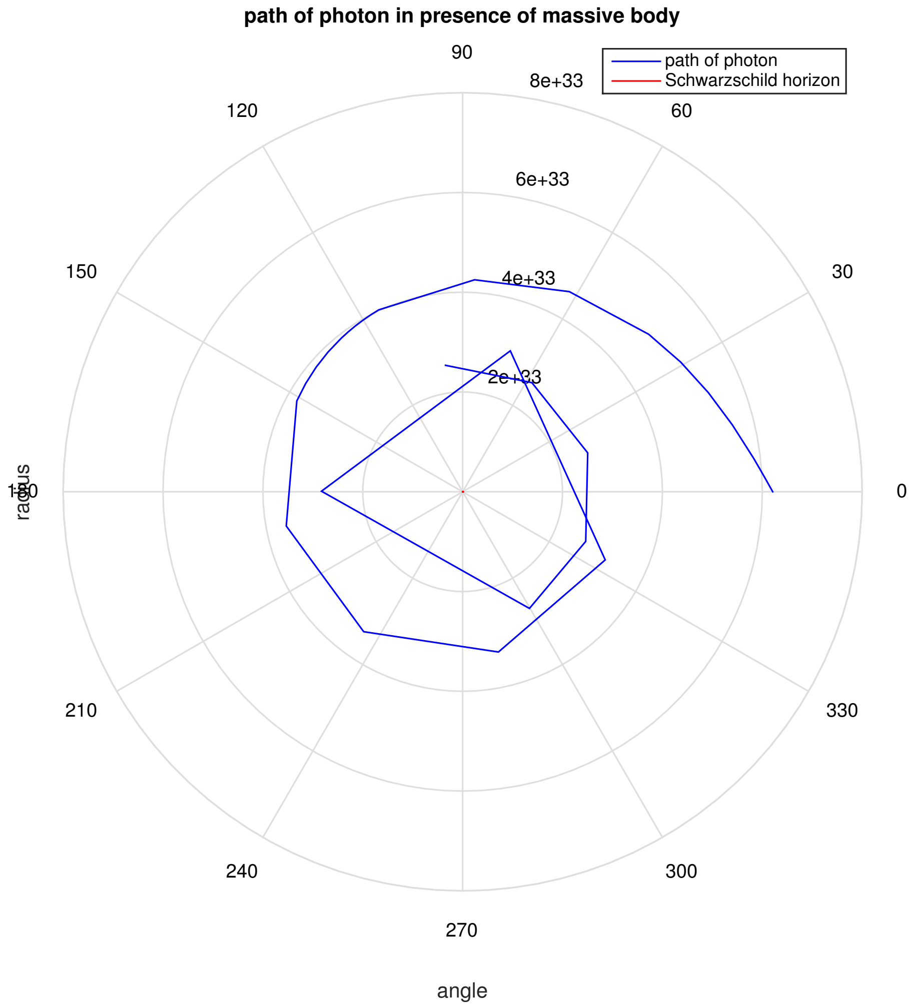

Figure 20: initial radius = 7 $\times 10^{35}$r$_s$, initial angle = 120 deg, running time = 1 $\times 10^{26}$

3.7.3 Discussion

From the figures above, we observe that a photon directed towards the Earth at 90 and 120 degrees, from a distance 7×10^6 to 7×10^{31} times the Schwarszchild radius, or 7×10^3m to 7×10^{27}m away from the surface of the earth, will be pulled in to the Earth. It will not actually be swallowed into the Schwarszchild radius, as illustrated in the images, since the *actual surface* of the Earth would inhibit that. However, if we were observing a black hole with the same properties, the photon would be pulled in to the center of the blackhole.

A photon directed towards the Earth at 10 and 40 degrees, from a distance 7×10^6 to

7×10^{13} times the Schwarszchild radius, or 7×10^{3}m to 7×10^{10}m away from the surface of the earth, will escape the pull of Earth's gravitational field.

We observe interesting motion in figures 20 and 21. In figure 20, a photon is directed towards the earth at 90 degrees, 7×10^{31}m away from the surface of the earth. It makes a circle around the Earth, which is pulled in to the center slightly. If the figure was observed for longer, such as 1×10^{26} T, the photon's observed motion becomes chaotic, crossing lines, but the MATLAB code does not determine that the photon is pulled in to the Earth.

In figure 21, a photon is directed towards the earth at 120 degrees, 7×10^{32}m away from the surface of the earth. We observe a spiralic, and the MATLAB code does not determine that the photon is pulled in to the Earth. Though at smaller radiuses, the 90 and 120 degree cases were non-escape examples, if the photons are initially far enough from a massive object, they may not be pulled in to the Schwarzschild radius.

shewn.

4 References

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